

On the Theory of the Capillary Tube.

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A recent paper by Richards and Coombs* discusses in some detail the determination of surface tension by the rise of the liquid in capillary tubes, and reflects mildly upon the inadequate assistance afforded by mathematics. It is true that no complete analytical solution of the problem can be obtained, even when the tube is accurately cylindrical. We may have recourse to graphical constructions, or to numerical calculations by the method of Runge,† who took an example from this very problem. But for experimental purposes all that is really needed is a sufficiently approximate treatment of the two extreme cases of a narrow and of a wide tube. The former question was successfully attacked by Poisson, whose final formula [(18) below] would meet all ordinary requirements. Unfortunately doubts have been thrown upon the correctness of Poisson's results, especially by Mathieu,‡ who rejects them altogether in the only case of much importance, *i.e.* when the liquid wets the walls of the tube—a matter which will be further considered later on. Mathieu also reproaches Poisson's investigation as implying two different values of h , of which the second is really only an improvement upon the first, arising from a further approximation. It must be admitted, however, that the problem is a delicate one, and that Poisson's explanation at a critical point leaves something to be desired. In the investigation which follows I hope to have succeeded in carrying the approximation a stage beyond that reached by Poisson.

In the theory of narrow tubes the lower level from which the height of the meniscus is reckoned is the free plane level. In experiment, the lower level is usually that of the liquid in a wide tube connected below with the narrow one, and the question arises how wide this tube needs to be in order that the inner part of the meniscus may be nearly enough plane. Careful experiments by Richards and Coombs led to the conclusion that in the case of water the diameter of the wide tube should exceed 33 mm., and that probably 38 mm. suffices. Such smaller diameters as are often employed (20 mm.) involve very appreciable error. Here, again, we should naturally look to mathematics to supply the desired information. The case of a straight

* 'Journ. Amer. Chem. Soc.,' No. 7, July, 1915.

† 'Math. Ann.,' vol. 46, p. 175 (1895).

‡ 'Théorie de la Capillarité,' Paris, 1883, pp. 46-49.

wall, making the problem two-dimensional, is easy,* but that of the circular wall is much more complicated.

Some drawings (from theory) given by Kelvin, figs. 24, 26, 28,† indicate clearly that diameters of 1·8 cm. and 2·6 cm. are quite inadequate. I have attempted below an analytical solution, based upon the assumption that the necessary diameter is large, as it will be, if the prescribed error at the axis is small enough. Although this assumption is scarcely justified in practice, the calculation indicates that a diameter of 4·7 cm. may not be too large.

As Richards and Coombs remark, the observed curvature of the lower part of the meniscus may be used as a test. Theory shows that there should be no sensible departure from straightness over a length of about 1 cm.

The Narrow Tube.

For the surface of liquid standing in a vertical tube of circular section, we have

$$x \sin \psi = \frac{x dz/dx}{\sqrt{\{1+(dz/dx)^2\}}} = \frac{1}{a^2} \int_0^x zx dx, \quad (1)$$

in which z is the vertical co-ordinate measured upwards from the free plane level, x is the horizontal co-ordinate measured from the axis, ψ is the angle the tangent at any point makes with the horizontal, and $a^2 = Tg\rho$,‡ where T is the surface-tension, g the acceleration of gravity, and ρ the density of the fluid. The equation expresses the equilibrium of the cylinder of liquid of radius x . At the wall, where $x = r$, ψ assumes a given value ($\frac{1}{2}\pi - i$), and (1) becomes

$$a^2 r \cos i = \int_0^r zx dx. \quad (2)$$

If the radius (r) of the tube is *small*, the total curvature is nearly constant, that is, the surface is nearly spherical. We take

$$z = l - \sqrt{(c^2 - x^2)} + u, \quad (3)$$

where l is the height of the centre and c the radius of the sphere, while u represents the correction required for a closer approximation. If we omit u altogether, (2) gives

$$a^2 r \cos i = \frac{1}{2}lr^2 + \frac{1}{3}\{(c^2 - r^2)^{3/2} - c^3\}. \quad (4)$$

* Compare 'Phil. Mag.,' vol. 34, p. 309, Appendix, 1892; 'Scientific Papers,' vol. 4, p. 13.

† The reference is given below.

‡ It may be remarked that a^2 is sometimes taken to denote the double of the above quantity.

Also, if h be the height at the lowest point of the meniscus, the quantity directly measured in experiment,

$$h = l - c. \quad (5)$$

In this approximation $r/c = \cos i$, and thus in terms of c

$$a^2 r^2 / c = \frac{1}{2} r^2 (h + c) + \frac{1}{3} (c^2 - r^2)^{3/2} - \frac{1}{3} c^3. \quad (6)$$

When the angle of contact (i) is zero, $c = r$, and

$$a^2 = \frac{1}{2} r (h + \frac{1}{3} r), \quad (7)$$

the well known formula.

When we include u , it becomes a question whether we should retain the value of c , *i.e.* $r \sec i$, appropriate when the surface is supposed to be exactly spherical. It appears, however, to be desirable, if not necessary, to leave the precise value of c open. Substituting the value of z from (3) in (1), we get, with neglect of $(du/dx)^3$,

$$1 + \frac{du}{dx} \frac{(c^2 - x^2)^{3/2}}{x c^2} - \frac{3(c^2 - x^2)^2}{2c^4} \left(\frac{du}{dx} \right)^2 = \frac{c}{a^2 x^2} \left[\frac{l x^2}{2} + \frac{(c^2 - x^2)^{3/2} - c^3}{3} + \int_0^x u x dx \right]. \quad (8)$$

For the purposes of the next approximation we may omit $(du/dx)^2$ and the integral, which is to be divided by a^2 . Thus

$$\frac{du}{dx} = \left(\frac{cl}{2a^2} - 1 \right) \frac{c^2 x}{(c^2 - x^2)^{3/2}} - \frac{c^6}{3a^2} \frac{1}{x(c^2 - x^2)^{3/2}} + \frac{c^3}{3a^2 x}, \quad (9)$$

and on integration

$$u = c^2 \left(\frac{cl}{2a^2} - \frac{c^2}{3a^2} - 1 \right) \frac{1}{\sqrt{(c^2 - x^2)}} + \frac{c^3}{3a^2} \log \{c + \sqrt{(c^2 - x^2)}\} + C. \quad (10)$$

We suppose with Poisson and Mathieu that

$$\frac{cl}{2a^2} - \frac{c^2}{3a^2} - 1 = 0, \quad (11)$$

so that

$$u = \frac{c^3}{3a^2} \log \{c + \sqrt{(c^2 - x^2)}\} + C, \quad (12)$$

corresponding to

$$\frac{du}{dx} = \frac{c^3}{3a^2} \frac{\sqrt{(c^2 - x^2)} - c}{x \sqrt{(c^2 - x^2)}}. \quad (13)$$

To determine c we have the boundary condition

$$\begin{aligned} \cot i &= \left(\frac{dz}{dx} \right)_{x=r} = \frac{r}{\sqrt{(c^2 - r^2)}} + \left(\frac{du}{dx} \right)_{x=r} \\ &= \frac{r}{\sqrt{(c^2 - r^2)}} \left\{ 1 - \frac{c^3}{3a^2} \frac{c - \sqrt{(c^2 - r^2)}}{r^2} \right\}, \end{aligned} \quad (14)$$

which gives c in terms of i and r . Explicitly

$$c = \frac{r}{\cos i} - \frac{r^3}{3a^2} \frac{\sin^2 i}{\cos^3 i (1 + \sin i)}. \quad (15)$$

These latter equations are given by Mathieu.

We have now to find the value of a^2 to the corresponding approximation. For the observed height of the meniscus

$$h = l - c + u_{x=0} = l - c + C + \frac{c^3}{3a^2} \log(2c); \quad (16)$$

and

$$\begin{aligned} a^2 r \cos i &= \int_0^r z x dx = \frac{r^2}{2} (l + C) + \frac{1}{3} \{ (c^2 + r^2)^{3/2} - c^3 \} + \int_0^r (u - C) x dx \\ &= (h + c) \frac{r^2}{2} + \frac{1}{3} \{ (c^2 - r^2)^{3/2} - c^3 \} \\ &\quad + \frac{c^3}{3a^2} \left[\frac{r^2}{2} \log \frac{c + \sqrt{(c^2 - r^2)}}{2c} + \frac{\{ c + \sqrt{(c^2 - r^2)} \}^2}{4} - c \sqrt{(c^2 - r^2)} \right]. \end{aligned} \quad (17)$$

In the important case where $i = 0$, the liquid *wetting* the walls of the tube, $c = r$ simply, and

$$\begin{aligned} a^2 &= \frac{r}{2} \left(h + \frac{r}{3} \right) - \frac{r^4}{6a^2} \left(\log 2 - \frac{1}{2} \right) \\ &= \frac{r}{2} \left\{ h + \frac{r}{3} - \frac{2r^2}{3h} \left(\log 2 - \frac{1}{2} \right) \right\} \\ &= \frac{1}{2} r \left(h + \frac{1}{3} r - 0.1288 r^2 / h \right). \end{aligned} \quad (18)$$

This is the formula given long since by Poisson,* the only difference being that his a^2 is the double of the quantity here so denoted.

It is remarkable that Mathieu rejects the above equations as applicable to the case $i = 0$, $c = r$, on the ground that then du/dx in (13) becomes infinite when $x = r$. But $d\sqrt{(r^2 - x^2)}/dx$, with which du/dx comes into comparison, is infinite at the same time; and, in fact, both

$$\frac{du}{dx} (r^2 - x^2)^{3/2} \quad \text{and} \quad \left(\frac{du}{dx} \right)^2 (r^2 - x^2)^2,$$

in equation (8) *vanish* when $x = r$. It is this circumstance which really determines the choice of l in (11).

We may now proceed to a yet closer approximation, introducing approximate values of the terms previously neglected altogether. From (13)

$$(c^2 - x^2)^2 \left(\frac{du}{dx} \right)^2 = \frac{c^6}{9a^4 x^2} \{ c^2 (c^2 - x^2) + (c^2 - x^2)^2 - 2c(c^2 - x^2)^{3/2} \},$$

and from (12)

$$\int_0^x u x dx = \frac{1}{2} C x^2 + \frac{c^3}{6a^2} [x^2 \log \{ c + \sqrt{(c^2 - x^2)} \} + \frac{1}{2} c^2 - c \sqrt{(c^2 - x^2)} + \frac{1}{2} (c^2 - x^2)].$$

* 'Nouvelle Théorie de l'Action Capillaire,' 1831, p. 112.

$$\begin{aligned}
\text{Thus } \frac{du}{dx} &= \left\{ \frac{c(l+C)}{2a^2} - 1 \right\} \frac{c^2x}{(c^2-x^2)^{3/2}} - \frac{c^6}{3a^2x(c^2-x^2)^{3/2}} + \frac{c^3}{3a^2x} \\
&\quad + \frac{c^4}{6xa^4(c^2-x^2)^{3/2}} \left[\frac{3c^2(c^2-x^2)}{2} + (c^2-x^2)^2 - 2c(c^2-x^2)^{3/2} \right. \\
&\quad \left. + c^2x^2 \log \{c + \sqrt{(c^2-x^2)}\} + \frac{c^4}{2} - c^3\sqrt{(c^2-x^2)} \right], \quad (19) \\
u &= \left\{ \frac{c(l+C)}{2a^2} - \frac{c^2}{3a^2} - 1 \right\} \frac{c^2}{\sqrt{(c^2-x^2)}} + \frac{c^3}{3a^2} \log \{c + \sqrt{(c^2-x^2)}\} \\
&\quad + \frac{c^5}{6a^4} \left[-2 \log \{c + \sqrt{(c^2-x^2)}\} + \frac{\sqrt{(c^2-x^2)}}{c} - 1 + \frac{c}{2\sqrt{(c^2-x^2)}} \right. \\
&\quad \left. + \frac{c}{\sqrt{(c^2-x^2)}} \log \{c + \sqrt{(c^2-x^2)}\} \right] + \text{constant.}
\end{aligned}$$

We have now to choose l , or rather $(l+C)$, and it may appear at first sight as though we might take it almost at pleasure. But this is not the case, at any rate if we wish our results to be applicable when $c=r$. For this purpose it is necessary that $(du/dx)_r \times (r^2-x^2)$ be a small quantity, and only a particular choice of $(l+C)$ will make it so. For when $x=c=r$,

$$\left(\frac{du}{dx} \right)_r \frac{r^2-x^2}{r^2} = \frac{r}{\sqrt{(r^2-x^2)}} \left\{ \frac{r(l+C)}{2a^2} - 1 - \frac{r^2}{3a^2} + \frac{r^4}{6a^4} \left(\log r + \frac{1}{2} \right) \right\} - \frac{r^4}{6a^4}$$

+ terms vanishing when $x=r$.

We must therefore take

$$\frac{c(l+C)}{2a^2} - 1 - \frac{c^2}{3a^2} + \frac{c^4}{6a^4} \left(\log c + \frac{1}{2} \right) = 0, \quad (20)$$

making

$$\begin{aligned}
u &= \frac{c^3}{3a^2} \left(1 - \frac{c^2}{a^2} \right) \log \frac{c + \sqrt{(c^2-x^2)}}{c} \\
&\quad + \frac{c^5}{6a^4} \left[\frac{\sqrt{(c^2-x^2)}-c}{c} + \frac{c}{\sqrt{(c^2-x^2)}} \log \frac{c + \sqrt{(c^2-x^2)}}{c} \right] + C'. \quad (21)
\end{aligned}$$

It should be noticed that u so determined does not become infinite when $c=r$ and $x=r$. For we have

$$u_r = -\frac{r^5}{6a^4} + \frac{r^6}{6a^4\sqrt{(r^2-x^2)}} \log \left\{ 1 + \frac{\sqrt{(r^2-x^2)}}{r} \right\} + C' = C'.$$

Also with the general value of c

$$u_0 = \frac{c^3}{3a^2} \left(1 - \frac{c^2}{a^2} \right) \log 2 + C'. \quad (22)$$

As before

$$h = l - c + u_0,$$

$$\begin{aligned}
 \text{and} \quad ra^2 \cos i &= \frac{(l+C')r^2}{2} + \frac{(c^2-r^2)^{3/2}-c^3}{3} + \int_0^r (u-C')x \, dx \\
 &= \frac{r^2}{2} \left\{ h+c - \frac{c^3}{3a^2} \left(1 - \frac{c^2}{2a^2} \right) \log 2 \right\} + \frac{(c^2-r^2)^{3/2}-c^3}{3} + \int_0^r (u-C')x \, dx. \quad (23)
 \end{aligned}$$

The integral in (23) can be expressed.

We find

$$\begin{aligned}
 \int_0^r (u-C')x \, dx &= \frac{c^3}{3a^2} \left(1 - \frac{c^2}{a^2} \right) \left[\frac{r^2}{2} \log \frac{c+\sqrt{(c^2-r^2)}}{c} + \frac{c^2}{4} - \frac{c\sqrt{(c^2-r^2)}}{2} + \frac{c^2-r^2}{4} \right] \\
 &+ \frac{c^5}{6a^4} \left[\frac{c^3-(c^2-r^2)^{3/2}}{3c} - \frac{r^2}{2} - c\{c+\sqrt{(c^2-r^2)}\} \left\{ \log \frac{c+\sqrt{(c^2-r^2)}}{c} - 1 \right\} \right. \\
 &\quad \left. + 2c^2(\log 2 - 1) \right]. \quad (24)
 \end{aligned}$$

The expression for $ra^2 \cos i$ in terms of c is complicated, and so is the relation between c and i demanded by the boundary condition

$$\cot i = \frac{r}{\sqrt{(c^2-r^2)}} + \left(\frac{du}{dx} \right)_{x=r}. \quad (25)$$

But in the particular case of greatest interest ($i = 0$) much simplification ensues. It follows easily from (25) that $c = r$. When we introduce this condition into (24), we get

$$\int_0^r (u-C')x \, dx = \frac{r^5}{12a^2} + \frac{r^7}{6a^4} \left(2 \log 2 - \frac{5}{3} \right), \quad (26)$$

and accordingly

$$a^2 = \frac{r}{2} \left(h + \frac{r}{3} \right) - \frac{r^4}{6a^2} \left(\log 2 - \frac{1}{2} \right) + \frac{5r^6}{36a^4} (3 \log 2 - 2). \quad (27)$$

Hence by successive approximations

$$\begin{aligned}
 a^2 &= \frac{r}{2} \left\{ h + \frac{r}{3} - \frac{2r^2}{3h} \left(\log 2 - \frac{1}{2} \right) + \frac{r^3}{9h^2} (32 \log 2 - 21) \right\} \\
 &= \frac{1}{2} r \left\{ h + \frac{1}{3} r - 0.1288 r^2/h + 0.1312 r^3/h^2 \right\}. \quad (28)
 \end{aligned}$$

If the ratio of r to h is at all such as should be employed in experiment, this formula will yield a^2 , viz., $T/g\rho$, with abundant accuracy.

Our equations give for the whole height of the meniscus in the case $i = 0$, $c = r$,

$$z_r - z_0 = r + u_r - u_0 = r - \frac{r^3}{3a^2} \left(1 - \frac{r^2}{2a^2} \right) \log 2. \quad (29)$$

Another method of calculating the correction for a small tube, originating apparently with Hagen and Desains, is to assume an elliptical form of surface in place of the circular, the minor axis of the ellipse being vertical. In any

case this should allow of a closer approximation, and drawings made for Kelvin* by Prof. Perry suggest that the representation is really a good one.

If the semi-axis minor of the ellipse be β , the curvature at the end of this axis is β/r^2 , and in our previous notation $\beta = hr^2/2a^2$. Also, i being equal to 0,

$$a^2 r = \int_0^r z x dx = \int_0^r \left\{ h + \beta - \beta \sqrt{1 - \frac{x^2}{\beta^2}} \right\} x dx,$$

$$\text{and} \quad a^2 = \frac{1}{2} r \left(h + \frac{1}{3} \beta \right) = \frac{1}{2} h r \left(1 + r^2/6a^2 \right). \quad (30)$$

This yields a quadratic in a^2 ; hence

$$\begin{aligned} a^2 &= \frac{hr}{4} + \frac{hr}{4} \sqrt{1 + \frac{4r}{3h}} = \frac{r}{2} \left\{ h + \frac{r}{3} - \frac{r^2}{9h} + \frac{2r^3}{27h^2} \right\} \\ &= \frac{1}{2} r \left\{ h + \frac{1}{3} r - 0.1111 r^2/h + 0.0741 r^3/h^2 \right\} \end{aligned} \quad (31)$$

approximately. It will be seen that this differs but little numerically from (28), which, however, professes to be the accurate result so far as the term in r^3/h^2 inclusive.

The Wide Tube.

The equation of the second order for the surface of the liquid, assumed to be of revolution about the axis of z , is well known and may be derived from (1) by differentiation. It is

$$\frac{d^2 z}{dx^2} + \frac{1}{x} \frac{dz}{dx} \left\{ 1 + \left(\frac{dz}{dx} \right)^2 \right\} = \frac{z}{a^2} \left\{ 1 + \left(\frac{dz}{dx} \right)^2 \right\}^{3/2} \quad (32)$$

If dz/dx be small, (32) becomes approximately

$$\frac{d^2 z}{dx^2} + \frac{1}{x} \frac{dz}{dx} - \frac{z}{a^2} = \frac{3z}{2a^2} \left(\frac{dz}{dx} \right)^2 - \frac{1}{x} \left(\frac{dz}{dx} \right)^3. \quad (33)$$

In the interior part of the surface under consideration $(dz/dx)^2$ may be neglected, and the approximate solution is

$$z = h_0 J_0(x/a) = h_0 I_0(x/a) = h_0 \left\{ 1 + \frac{x^2}{2^2 a^2} + \frac{x^4}{2^2 \cdot 4^2 \cdot a^2} + \dots \right\}, \quad (34)$$

J_0 denoting, as usual, the Bessel's, or rather Fourier's, function of zero order, and h_0 being the elevation at the axis above the free absolutely plane level. For the present purpose h_0 is to be so small as to be negligible in experiment, and the question is how large must r be.

When h_0 is small enough, x/a may be large while dz/dx still remains small. Eventually dz/dx increases so that the formula fails. But when x is large enough before this occurs, we may if necessary carry on with the two-dimensional solution properly adjusted to fit, as will be further explained

* 'Proc. Roy. Inst.,' 1886; 'Popular Lectures and Addresses,' I, p. 40.

later. In the meantime it will be convenient to give some numerical examples of the increase in dz/dx . In the usual notation

$$\frac{dz}{dx} = \frac{h_0}{a} I_1\left(\frac{x}{a}\right), \quad (35)$$

and the values of I_1 , up to $x/a = 6$, are tabulated.*

In the case of water $a = 0.27$ cm. If we take $h_0/a = 0.01$, and $x/a = 4$, we have $dz/dx = 0.098$, so that $(dz/dx)^2$ is still fairly small. Here for water $h_0 = 0.0027$ cm. and $2x = 2.2$ cm. A diameter of 2.2 cm. is thus quite insufficient, unless an error exceeding 0.003 cm. be admissible. Again, suppose $h_0/a = 0.001$, and take $x/a = 6$. Then $dz/dx = 0.061$, again small. For water $h_0 = 0.00027$ cm., and $2x = 3.2$ cm. This last value of h_0 is about that (0.003 mm.) given by Richards and Coombs as the maximum admissible error of reading, and we may conclude that a diameter of 3.2 cm. is quite inadequate to take advantage of this degree of refinement.

We may go further in this example without too great a loss of accuracy. Retaining $h_0/a = 0.001$, let us make $x/a = 7$. I find $I_1(7) = 156$, about, so that the extreme value of dz/dx is 0.156, still moderately small. Here $2x = 3.8$ cm., which is thus shown to be inadequate in the case of water.

But apart from the question of the necessary diameter of tube, information sufficient for experimental purposes can be derived in another manner. The initial value of z (on the axis) is h_0 ; and $z = 2h_0$ when $I_0(x/a) = 2$, *i.e.* when $x = 1.8a$. For the best work h_0 should be on the limit of what can be detected and then h_0 and $2h_0$ could just be distinguished. The observer may be satisfied if no difference of level can be seen over the range $x = \pm 1.8a$; in the case of water this range is $2 \times 1.8 \times 0.27 = 0.97$ cm., or say 1 cm.

It has already been remarked that when h_0 is small enough x/a may become great within the limits of application of (35). To shorten our expressions we will take a temporarily as the unit of length. Then when x is very great,

$$I_1(x) = I_0(x) = \frac{e^x}{\sqrt{(2\pi x)}}. \quad (36)$$

Thus if ψ be the angle the tangent to the curve makes with the horizontal,

$$\psi = \tan \psi = \frac{dz}{dx} = \frac{h_0 e^x}{\sqrt{(2\pi x)}}; \quad (37)$$

an equation which may be employed when h_0 is so small that a large x is consistent with a small ψ .

In order to follow the curve further, up to $\psi = \frac{1}{2}\pi$, we may employ the two-dimensional solution, the assumption being that the region of

* Brit. Assoc. Rep. for 1889'; or Gray and Mathews' 'Bessel's Functions,' Table VI.

moderate ψ occupies a range of x small in comparison with its actual value, *i.e.* a value not much less than r , the radius of the tube. On account of the magnitude of x we have only the one curvature to deal with. For this curvature

$$\frac{1}{R} = \frac{d\psi}{ds} = \frac{d\psi}{dz} \sin \psi = z, \quad (38)$$

so that

$$\frac{1}{2}z^2 = C - \cos \psi = 1 - \cos \psi,$$

since when $\psi = 0$, z^2 is exceedingly small. Accordingly

$$z = 2 \sin \frac{1}{2} \psi. \quad (39)$$

$$\text{Also} \quad dx = \frac{dz}{\tan \psi} = \left(\frac{1}{\sin \frac{1}{2} \psi} - 2 \sin \frac{1}{2} \psi \right) d\left(\frac{1}{2} \psi\right),$$

and

$$x = \log \tan \left(\frac{1}{4} \psi\right) + 2 \cos \frac{1}{2} \psi + C'. \quad (40)$$

The constant is determined by the consideration that at the wall ($x = r$), $\psi = \frac{1}{2}\pi$; thus

$$\begin{aligned} r - x &= \log \tan (\pi/8) + \sqrt{2} - \log \tan \left(\frac{1}{4} \psi\right) - 2 \cos \left(\frac{1}{2} \psi\right) \\ &= \log \tan (\pi/8) + \sqrt{2} - 2 + 2 \log 2 - \log \psi, \end{aligned} \quad (41)$$

since ψ is small.

The value of x is supposed to be the same here as in (37), so that

$$x = \log \psi + \frac{1}{2} \log (2\pi x) - \log h_0, \quad (42)$$

whence on elimination of ψ and restoration of a ,

$$r/a = -\log (\sqrt{2} + 1) + \sqrt{2} - 2 + 2 \log 2 + \frac{1}{2} \log (2\pi x/a) - \log (h_0/a). \quad (43)$$

With sufficient approximation, when h_0 is small enough, we may here substitute r for x , and thus

$$\begin{aligned} r/a - \frac{1}{2} \log (r/a) &= -\log (\sqrt{2} + 1) + \sqrt{2} - 2 + 2 \log 2 + \frac{1}{2} \log (2\pi) - \log (h_0/a) \\ &= 0.8381 + \log (a/h_0). \end{aligned} \quad (44)$$

This formula should give the relation between r/a and h_0/a when h_0/a is small enough, but it is only roughly applicable to the case of greatest interest, where $a/h_0 = 1000$, corresponding to the accuracy of reading found by Richards and Coombs. In this case

$$0.8381 + \log (a/h_0) = 7.746.$$

For this value of r/a we should have $\frac{1}{2} \log (r/a) = 1.024$. It is true that according to (44) r/a will be somewhat greater, but on the other hand the proper value of x (replaced by r) is less than r . We may fairly take

$$r/a = 7.746 + 1.024 = 8.770,$$

making with $a = 0.27$ cm.

$$2r = 4.74 \text{ cm.}$$

This calculation indicates that a diameter greater even than those con-

templated by Richards and Coombs may be necessary to reduce h_0 to negligibility, but it must be admitted that it is too rough to inspire great confidence in the close accuracy of the final number. Probably it would be feasible to continue the approximation, employing an approximate value for the second curvature in place of neglecting it altogether. But although the integration can be effected, the work is rather long.

[*Added November 17.*—Since this paper was communicated, I have been surprised to find that the problem of the last paragraphs was treated long ago by Laplace in the ‘*Mécanique Céleste*’* by a similar method, and with a result equivalent to that (44) arrived at above for the relation between the radius of a wide tube and the small elevation at the axis. Laplace uses the definite integral expression for I_0 , and obtains the approximate form appropriate to large arguments. In view of Laplace’s result, I have been tempted to carry the approximation further, as suggested already.

In the previous notation, the differential equation of the surface may be written

$$\frac{\sin \psi d\psi}{dz} + \frac{\sin \psi}{x} = \frac{z}{a^2}. \quad (45)$$

In the first approximation, where the second curvature on the left is omitted, we get

$$\frac{z^2 - z_0^2}{2a^2} = 1 - \cos \psi = 2 \sin^2 \frac{\psi}{2},$$

z_0 being the elevation at the axis, where $\psi = 0$. For the present purpose z_0^2 is to be regarded as exceedingly small, so that we may take at this stage, as in (39),

$$z = 2a \sin \left(\frac{1}{2} \psi \right). \quad (46)$$

We now introduce an approximate value for the second curvature in (45), writing $x = r$, where r is the radius of the tube, and making, according to (46),

$$\sin \psi = \frac{z}{a} \sqrt{1 - \frac{z^2}{4a^2}}. \quad (47)$$

On integration

$$C - \cos \psi = \frac{z^2}{2a^2} + \frac{4a}{3r} \left(1 - \frac{z^2}{4a^2} \right)^{3/2} = \frac{z^2}{2a^2} + \frac{4a}{3r} \cos^3 \frac{\psi}{2}, \quad (48)$$

on substitution in the small term of the approximate value of z . When $\psi = 0$, z^2 is very small, so that $C = 4a/3r$, and

$$\frac{z}{a} = 2 \sin \frac{\psi}{2} + \frac{2a}{3r} \frac{1 - \cos^3 \left(\frac{1}{2} \psi \right)}{\sin \frac{1}{2} \psi}, \quad (49)$$

is the second approximation to z .

* Supplément au X^e Livre, pp. 60–64, 1805.

From (49)

$$\frac{1}{a} \frac{dz}{d\psi} = \cos \frac{\psi}{2} + \frac{a}{3r} \frac{3 \cos^2 \frac{1}{2}\psi \cdot \sin^2 \frac{1}{2}\psi - \cos \frac{1}{2}\psi (1 - \cos^3 \frac{1}{2}\psi)}{\sin^2 (\frac{1}{2}\psi)}. \quad (50)$$

We are now in a position to find x by the relation

$$x = \int \cot \psi (dz/d\psi) d\psi, \quad (51)$$

the constant of integration being determined by the correspondence of $x = r$, $\psi = \frac{1}{2}\pi$. Thus

$$\begin{aligned} \frac{r-x}{a} = & \log(\sqrt{2}-1) + \sqrt{2} + \frac{a}{3r} \left\{ -2 + \frac{\sqrt{2}}{2} + \frac{3}{2} \log(\sqrt{2}-1) \right\} - \log \tan \frac{\psi}{4} \\ & - 2 \cos \frac{\psi}{2} + \frac{a}{3r} \left\{ \frac{1}{2(1-\cos \frac{1}{2}\psi)} + 2 \sin^2 \frac{\psi}{2} - \frac{3}{2} \log \frac{1-\cos \frac{1}{2}\psi}{\sin \frac{1}{2}\psi} \right\}, \end{aligned} \quad (52)$$

giving when ψ is small

$$\frac{r-x}{a} = \alpha - \frac{a\beta}{3r} - \left(1 + \frac{a}{2r}\right) \log \psi, \quad (53)$$

$$\text{where} \quad \alpha = \log(\sqrt{2}-1) + \sqrt{2} + \log 4 - 2 = -0.0809, \quad (54)$$

$$-\frac{1}{3}\beta = \log 2 + \frac{1}{2} \log(\sqrt{2}-1) + \frac{1}{6}\sqrt{2} - 7/12 = -0.0952. \quad (55)$$

The other equation, derived from the flat part of the surface, is

$$\psi = \frac{dz}{dx} = \frac{h_0}{a} I_1 \left(\frac{x}{a} \right) = \frac{h_0/a \cdot e^{x/a}}{\sqrt{(2\pi x/a)}} \left(1 - \frac{3a}{8x} \right), \quad (56)$$

in which x/a is regarded as large; or

$$\frac{x}{a} = \log \psi + \log \frac{a}{h_0} + \frac{1}{2} \log \frac{2\pi x}{a} + \frac{3a}{8x}. \quad (57)$$

In equations (53), (57) x and ψ are to be identified. On elimination of ψ

$$\frac{r}{a} - \log \frac{a}{h_0} = \frac{\alpha - a\beta/3r}{1 + a/2r} + \frac{r-x}{2r+a} + \frac{1}{2} \log \frac{2\pi x}{a} + \frac{3a}{8x}, \quad (58)$$

in which we may put

$$\begin{aligned} \log \frac{2\pi x}{a} &= \log \frac{2\pi r}{a} + \log \left(1 - \frac{r-x}{r} \right) = \log \frac{2\pi r}{a} - \frac{r-x}{r}, \\ \frac{3a}{8x} &= \frac{3a}{8r} \left(1 + \frac{r-x}{r} \right). \end{aligned}$$

$$\text{Thus} \quad \frac{r}{a} - \log \frac{a}{h_0} = \frac{\alpha - a\beta/3r}{1 + a/2r} + \frac{a(r-x)}{8r^2} + \frac{3a}{8r} + \frac{1}{2} \log \frac{2\pi r}{a}, \quad (59)$$

in which, since x is nearly equal to r , $a(r-x)/8r^2$ may usually be neglected. Also, in view of the smallness of α and β , it is scarcely necessary to retain the denominator $1 + a/2r$, so that we may write

$$\begin{aligned} \frac{r}{a} - \log \frac{a}{h_0} &= -0.0809 + 0.2798 \frac{a}{r} + \frac{1}{2} \log \frac{2\pi r}{a} \\ &= 0.8381 + 0.2798 a/r + \frac{1}{2} \log (r/a). \end{aligned} \quad (60)$$

The effect of the second approximation is the introduction of the second term on the right of (60).

To take an example, let us suppose as before that $a/h_0 = 1000$, so that $\log(a/h_0) = 6.908$. By successive approximation we find from (60)

$$r/a = 8.689, \quad (61)$$

so that if $a = 0.27$ cm. (as for water),

$$2r = 4.79 \text{ cm.} \quad (62)$$

The correction to Laplace's formula is here unimportant.

The above is the diameter of tube required to render h_0 negligible according to the standard adopted.

It may sometimes be convenient to invert the calculation, and deduce the value of h_0 from the diameter of the tube (not much less than 4 cm.) and an approximate value of a . For this purpose we may use (60), or preferably (59), taking $x = \frac{4}{3}r$ for instance. The calculated value of h_0 would then be used as a correction. The accompanying small Table may be useful for this purpose.

r/a .	$-\log_{10}(h_0/a)$.	Difference.	h_0/a .
6	1.8275		0.0149
7	2.2319	0.4044	0.0059
8	2.6399	0.4080	0.0023
9	3.0508	0.4109	0.00089
10	3.4639	0.4131	0.00034

We have supposed throughout that the liquid surface is symmetrical about the axis, as happens when the section of the containing tube is circular. It may be worth remarking that without any restriction to symmetry the differential equation of the nearly flat parts of a large surface may be taken to be

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} - \frac{z}{a^2} = 0, \quad (63)$$

so that z may be expressed by the series

$$z = A_0 I_0(r/a) + (A_1 \cos \theta + B_1 \sin \theta) I_1(r/a) + (A_2 \cos 2\theta + B_2 \sin 2\theta) I_2(r/a) + \dots \quad (64)$$

r, θ denoting the usual polar co-ordinates in the horizontal plane.]